Generics in a measurable wreath product

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Section 1

Polish groups and automatic continuity

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-Many other examples (some for later).

Automatic Continuity

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Answer: Might be hard, even if both groups are Polish. Consider $(\mathbb{R}, +)$ and $(\mathbb{R}^2, +)$.

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1. G has the Rohklin property if there is $g \in G$ such that g^G is dense.
2. G has the strong Rohklin property if there is $g \in G$ such that g^G is comeager.
3. G has ample generics if for any $n \in \mathbb{N}$ there is $(g_0, \ldots, g_{n-1}) \in G^n$ such that $(g_0, \ldots, g_{n-1})^G$ is comeager.

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The automorphism group of $(Q,<)$: SRP but no ample generics.

The automorphism group of $\mathbb{N}$: This is S_∞. It has ample generics.

Why we care about dynamics?

Theorem (Kechris-Rosendal)

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They also characterized when automorphism groups of countable models had any of the previous properties.

Section 2

A model-theoretic motivation

Randomization

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For a separable model $(M, d, \ldots)$, its Borel randomization is

$$M^R := (L^0([0,1], M), d^0, \ldots)$$

where

$$L^0([0,1], M) := \{f : [0,1] \to M | f \text{ is measurable}\}$$

And $d^0(f,g) := \int d(f(\omega), g(\omega)) d\lambda(\omega)$

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1. *(C.C. Moore) If M is Polish, then $L^0([0,1], M)$ is Polish.*
2. *(Ben Yaacov-Keisler) If M is ω-categorical, then M^R is ω-categorical.*
3. *(Ben Yaacov-Keisler) If M is stable, then M^R is stable.*

Automorphism group of M^R

Theorem (Ibarlucia)

Let $G = Aut(M)$, the group of isometries of a separable structure M. Then $Aut(M^R) = G \wr [0,1] := L^0([0,1], G) \rtimes Aut[0,1]$

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Fact: If G is a Polish group, then $L^0([0,1], G)$ and $\tilde{G}$ are also Polish groups.

Section 3

A topological motivation

Transfer properties of L_0

Theorem (Kaïchouh-Le Maitre)

Let G be a Polish group. If G has RP (resp. SRP, ample generics), then $L_0([0,1], G)$ also has RP (resp. SRP, ample generics).

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The automorphism group of a Randomization

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Proof.

Aut$[0,1]$ does not have ample generics. So by using Kuratowski-Ulam, one obtains a contradiction if $\tilde{G}$ has ample generics.

□

Rohklin property of $\tilde{G}$

Theorem (Berenstein, Z.)

If G is a Polish group with RP, then $\tilde{G}$ has the RP.

Section 5

Metric generics

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Let $G = \mathrm{ISO}(M)$:

- it has *metric generics* if it has an orbit whose uniform closure is comeager in the pointwise convergence topology;
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Examples(Ben Yacoov, Berenstein, Melleray): $\text{Aut}[0,1]$, the isometry group of the bounded Urysohn space, the unitary group of a separable Hilbert space.

Uniform automatic continuity

Theorem (Ben Yaacov, Berenstein, Melleray)

Let $G = ISO(M)$ with ample metric generics and $\varphi : G \to H$ a homomorphism for H separable topological group. If φ is continuous in the uniform topology, it is continuous in the pointwise convergence topology.

Metric generics in $\tilde{G}$

Theorem (Berenstein, Z.)

- *If G is a Polish group with metric generics, then $L^0([0,1],G)$ also has metric generics.*
- *If G is a Polish group with ample metric generics, then $L^0([0,1],G)$ also has ample metric generics.*

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Open Question: Does $\tilde{G}$ has automatic continuity if G does?

Thanks